

Estimating the output entropy of a tensor product of two quantum channels

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Abstract

In this paper we find, for a class of bipartite quantum states, a nontrivial lower bound on the entropy gain resulting from the action of a tensor product of identity channel with an arbitrary channel. By means of that we then estimate (from below) the output entropy of the tensor product of *dephasing* channel with an arbitrary channel. Finally, we provide a characterization of all *phase-damping* channels resulting as particular cases of *dephasing* channels.

1 Introduction

One of the most important tasks in quantum information theory is to calculate how informational capacities are changed under the external action on the quantum system. Most of used capacities are determined on the base of the von Neumann entropy. We consider the problem of finding a lower bound on the entropy gain for a state of bipartite system in the case if the perturbation affects only its subsystem. The nontriviality of this problem is connected with a phenomenon of entanglement for bipartite quantum systems which doesn't exist for classical systems.

Let us consider a state $\rho \in \mathfrak{S}(K)$, where $\mathfrak{S}(K)$ denotes the set of all positive unit-trace operators in the Hilbert space K^1 . Then, given a quantum channel (a linear completely positive trace-preserving map) Ω on the set of all bounded operators $B(K)$ on K , one can consider the entropy gain with respect to the action of a such channel

$$S(\Omega(\rho)) - S(\rho). \quad (1)$$

¹Where not explicitly said, the Hilbert spaces are assumed to be infinite dimensional.

Here S stands for the von Neumann entropy, $S(\rho) = -\text{Tr}(\rho \log \rho)$ with $0 \leq S(\rho) \leq +\infty$. The entropy gain is relevant since it measures the mixing property of the quantum channel [1].

Let us consider the Kraus decomposition of Ω

$$\Omega(\rho) = \sum_{j=1}^{+\infty} V_j \rho V_j^*, \quad (2)$$

then if there exists

$$s - \lim_{N \rightarrow +\infty} \sum_{j=1}^N V_j V_j^* = \Omega(I),$$

and if $S(\rho) < +\infty$, the following lower bound for the entropy gain (1) has been found [5]

$$S(\Omega(\rho)) - S(\rho) \geq -\text{Tr}(\rho \log \Omega(I)) \geq 0. \quad (3)$$

Moving to the context of bipartite systems, it is of interest to evaluate the entropy gain for a state $\rho \in \mathfrak{S}(H \otimes K)$ under the action of the tensor product of identity channel $Id : B(H) \rightarrow B(H)$ and the channel Ω , i.e.

$$S((Id \otimes \Omega)(\rho)) - S(\rho). \quad (4)$$

Let us consider the convex closure S_Ω of the output entropy for the state $\rho \in \mathfrak{S}(K)$ defined in [8] as

$$S_\Omega(\rho) = \inf_{\rho = \sum \pi_j \rho_j} \sum_j \pi_j S(\Omega(\rho_j)). \quad (5)$$

We conjecture the following relation between the two quantities (4) and (5)

Conjecture.

$$S((Id \otimes \Omega)(\rho)) - S(\rho) \geq S_\Omega(\text{Tr}_H(\rho)). \quad (6)$$

In this paper we shall find a sufficiently broad class of states ρ for which (6) holds true.

It is known that if the inequality (6) is valid for all states of the form $\rho = (\Phi \otimes Id)(\sigma)$, $\sigma \in \mathfrak{S}(H \otimes K)$, then the minimal output entropy would be additive with respect to tensor product of Φ . Proving this property was one of the motivation to also address the Conjecture (see [10]). It should be noticed that states of the form $(\Phi \otimes Id)(|e\rangle\langle e|)$ are studied in [11] in the other context.

The quantum channel Φ is said to be *dephasing* [4, 6] if there exists the orthonormal basis (e_n) in H such that

$$\Phi(|e_n\rangle\langle e_m|) = \lambda_{nm}|e_n\rangle\langle e_m|, \quad (7)$$

where (λ_{nm}) is a positive definite matrix with $\lambda_{nn} = 1$. Quantum dephasing channels are known to be complementary to entanglement-breaking channels [6].

We shall show that (6) is related to the following inequality

$$S((\Phi \otimes \Omega)(|e\rangle\langle e|)) \geq S((\Phi \otimes Id)(|e\rangle\langle e|)) + \sum_n \pi_n S(\Omega(|h_n\rangle\langle h_n|)), \quad (8)$$

where

$$|e\rangle\langle e| = \sum_{n,m} \lambda_n \bar{\lambda}_m |e_n\rangle\langle e_m| \otimes |h_n\rangle\langle h_m|,$$

with orthonormal system (e_n) in H , unit vectors $h_n \in K$, Φ is a dephasing channel while Ω is an arbitrary channel. The inequality (8) is closely related to the property of the strong superadditivity for the channel Φ introduced in [8] and widely discussed in [3].

If $H = L^2(\mathbb{R})$ the notion of a dephasing channel can be extended to the channel Φ for which the output state $\Phi(|\psi\rangle\langle\psi|)$ is the integral operator of the form [7]

$$(\Phi(|\psi\rangle\langle\psi|)\phi)(x) = \int_{\mathbb{R}} \lambda(x, y) \psi(x) \bar{\psi}(y) \phi(y) dy, \quad (9)$$

where $\phi \in L^2(\mathbb{R})$ and $\lambda(x, y)$ is a positive definite kernel. We shall call (9) a *generalised dephasing* channel. Using the generalised eigenvectors of the position operator we can represent the action of (9) in the form

$$\Phi(|x\rangle\langle y|) = \lambda(x, y)|x\rangle\langle y|. \quad (10)$$

Equation (10) should be understood in the sense of (9).

Let us consider a pure state

$$|e\rangle\langle e| = \int_{\mathbb{R}^2} \lambda(x) \bar{\lambda}(y) |x\rangle\langle y| \otimes |h_x\rangle\langle h_y| dx dy, \quad (11)$$

where $\int_{\mathbb{R}} |\lambda(x)|^2 dx = 1$, $x \rightarrow h_x$ is a measurable function and h_x are unit vectors in K . More precisely (11) means that for a scalar product of the

bipartite system

$$\langle \phi_1 \otimes \psi_1 | e \rangle \langle e | \phi_2 \otimes \psi_2 \rangle = \int_{\mathbb{R}^2} \lambda(x) \bar{\lambda}(y) \bar{\phi}_1(x) \phi_2(y) \langle \psi_1 | h_x \rangle \langle h_y | \psi_2 \rangle dx dy,$$

where $\phi_1, \phi_2 \in H$ and $\psi_1, \psi_2 \in K$. Similarly to (8) we shall show that the following inequality holds true

$$S((\Phi \otimes \Omega)(|e\rangle\langle e|)) \geq S((\Phi \otimes Id)(|e\rangle\langle e|)) + \int_{\mathbb{R}} \pi(x) S(\Omega(|h_x\rangle\langle h_x|)) dx \quad (12)$$

with $\pi(x) = |\lambda(x)|^2$, where Φ a generalised dephasing channel and Ω an arbitrary channel.

The quantum dephasing channel (7) is said to be a *phase-damping* channel if

$$\lambda_{nm} = \lambda_{n-m}, \quad \bar{\lambda}_n = \lambda_{-n}, \quad (13)$$

for some complex numbers $(\lambda_n)_{n=0}^{N-1}$ which are the discrete Fourier transform of a probability distribution $(\pi_n)_{n=0}^{N-1}$ [2]. In [2] the inequality (8) was obtained for a phase-damping channel Φ in the case $\dim H < +\infty$. We shall extend the definition of the phase-damping channel to the infinite-dimensional space. Then, based upon Bochner's theorem [9] we shall give a complete classification of all phase-damping channels.

The paper is organized as follows. In Section 2 some results about the entropy gain will be proved. As an application the inequalities of the form (8) and (12) are obtained for all dephasing channels. Section 3 is devoted to the description of quantum phase-damping channels of the form (13) as well as to their generalizations. The last Section contains concluding remarks.

2 The entropy gain

We first derive a tighter bound on (3) where the role of the identity operator I is played by the orthogonal projection P with the property $\text{supp} \rho \subset \text{supp} P$.

Proposition 1. *Suppose that for $\rho \in \mathfrak{S}(H)$, $S(\rho) < +\infty$, there is the orthogonal projection P such that*

$$\text{supp} \rho \subset \text{supp} P,$$

and the strong limit

$$s - \lim_{N \rightarrow \infty} \sum_{j=1}^N V_j P V_j^* = \Omega(P),$$

exists. Then, it is

$$S(\Omega(\rho)) - S(\rho) \geq -Tr(\Omega(\rho) \log \Omega(P)).$$

Proof. We shall follow the techniques of [5]. Let us pick up an orthonormal basis (e_k) spanning $supp P$ such that the state ρ , $supp \rho \subset supp P$, can be represented in the form

$$\rho = \sum_k \nu_k |e_k \rangle \langle e_k|.$$

Under the condition of Proposition 1,

$$S(\rho) = -Tr(\rho \log \rho) = \sum_k \nu_k (-\log \nu_k) < +\infty.$$

Then, there exists an operator F satisfying

$$Tr(\rho F) < +\infty, \quad Tr(\exp(-\beta F)) < +\infty, \quad \beta > 0. \quad (14)$$

Indeed, it suffices to set

$$F = \sum_k \mu_k (-\log \nu_k) |e_k \rangle \langle e_k|,$$

where (μ_k) are taken in such a way that $\mu_k \uparrow +\infty$ but $\sum_k \mu_k \nu_k (-\log \nu_k) < +\infty$ still converges. This allows us to define a state ρ_β as follows

$$\rho_\beta = \frac{P \exp(-\beta F)}{Tr(P \exp(-\beta F))}. \quad (15)$$

The monotonicity of the quantum relative entropy $S(\rho \parallel \sigma) = Tr(\rho(\log \rho - \log \sigma))$ gives

$$S(\Omega(\rho) \parallel \Omega(\rho_\beta)) \leq S(\rho \parallel \rho_\beta). \quad (16)$$

On the other hand,

$$S(\rho \parallel \rho_\beta) = -S(\rho) + \beta Tr(\rho F) + \log Tr(P \exp(-\beta F)),$$

then (16) implies that

$$Tr(\Omega(\rho)(-\log \Omega(\rho_\beta))) \leq S(\Omega(\rho)) - S(\rho) + \beta Tr(\rho F) + \log Tr(P \exp(-\beta F)). \quad (17)$$

Substituting (15) in (17) we get

$$S(\Omega(\rho)) - S(\rho) \geq Tr(\Omega(\rho)(-\log \Omega(P \exp(-\beta F)))) - \beta Tr(\rho F).$$

Since $P \exp(-\beta F) \leq P$ it implies $\log \Phi(P \exp(-\beta F)) \leq \log \Phi(P)$ by the operator monotonicity of the function $\log x$ on \mathbb{R}_+ . Taking the limit $\beta \rightarrow 0$ we obtain the desired result.

We now move on by considering a bipartite system $H \otimes K$ where Proposition 1 allows us to prove the following theorem.

Theorem 1. *Suppose that $\rho \in \mathfrak{S}(H \otimes K)$ has the form*

$$\rho = \sum_{n,m} \lambda_{mn} |e_n \rangle \langle e_m| \otimes |h_n \rangle \langle h_m|, \quad (18)$$

with orthonormal basis (e_n) in H , unit vectors $h_n \in K$ and a positive definite matrix (λ_{mn}) . Then,

$$S((Id \otimes \Omega)(\rho)) \geq S(\rho) + \sum_n \pi_n S(\Omega(|h_n \rangle \langle h_n|)),$$

where

$$\pi_n = \lambda_{nn},$$

and Ω is an arbitrary quantum channel.

Proof. Suppose that a state $\rho \in \mathfrak{S}(H \otimes K)$ has the form (20). Following [2] let us define the orthogonal projection P by the formula

$$P = \sum_n |e_n \rangle \langle e_n| \otimes |h_n \rangle \langle h_n|. \quad (19)$$

As consequence

$$P\rho = \rho P = \rho,$$

and hence

$$\text{supp} \rho \subset \text{supp} P.$$

Applying Proposition 1 to the state ρ and the channel $\Phi = Id \otimes \Omega$ we get

$$S((Id \otimes \Omega)(\rho)) - S(\rho) \geq -\text{Tr}((Id \otimes \Omega)(\rho) \log(Id \otimes \Omega)(P)).$$

Then using (19) the r.h.s. above results

$$\begin{aligned} & \text{Tr}((Id \otimes \Omega)(\rho) \log(Id \otimes \Omega)(P)) = \\ & \text{Tr}\left(\sum_{n,m} \lambda_{nm} |e_n \rangle \langle e_m| \otimes \Omega(|h_n \rangle \langle h_m|) \sum_k |e_k \rangle \langle e_k| \otimes \log(\Omega(|h_k \rangle \langle h_k|))\right) = \\ & \text{Tr}\left(\sum_n \lambda_{nn} \Omega(|h_n \rangle \langle h_n|) \log(\Omega(|h_n \rangle \langle h_n|))\right). \end{aligned}$$

Corollary 1. Suppose that $\rho \in \mathfrak{S}(H \otimes K)$ has the form

$$\rho = \sum_{n,m} \lambda_{mn} |e_n \rangle \langle e_m| \otimes |h_n \rangle \langle h_m|, \quad (20)$$

with orthonormal basis (e_n) in H , unit vectors $h_n \in K$ and a positive definite matrix (λ_{mn}) . Then,

$$S((Id \otimes \Omega)(\rho)) - S(\rho) \geq S_\Omega(Tr_H(\rho)).$$

Proof. It immediately follows from the inequality

$$\sum_n \pi_n S(\Omega(|h_n \rangle \langle h_n|)) \geq S_\Omega(\sigma),$$

where $\sigma = \sum_n \pi_n |h_n \rangle \langle h_n|$.

Corollary 2. The relation (8) holds for the dephasing channel Φ acting as

$$\Phi(|e_n \rangle \langle e_m|) = \lambda_{nm} |e_n \rangle \langle e_m|.$$

Proof. Given a unit vector $e \in H \otimes K$ and an orthonormal basis (e_n) in H there exist the unit vectors $h_n \in K$ and the complex numbers ν_n , $\sum_n |\nu_n|^2 = 1$, such that

$$|e \rangle \langle e| = \sum_{n,m} \nu_n \bar{\nu}_m |e_n \rangle \langle e_m| \otimes |h_n \rangle \langle h_m|.$$

It results in the state

$$\rho = (\Phi \otimes Id)(|e \rangle \langle e|) = \sum_{n,m} \lambda_{nm} \nu_n \bar{\nu}_m |e_n \rangle \langle e_m| \otimes |h_n \rangle \langle h_m|,$$

satisfying the conditions of Theorem 1 if we replace λ_{nm} by $\lambda_{nm} \nu_n \bar{\nu}_m$. Then the result follows.

Now, let us set $H = L^2(\mathbb{R})$ and consider the generalized eigenvectors $|x \rangle$ of the position operator \hat{x} acting on H as

$$(\hat{x}f)(x) = xf(x),$$

with $f(\cdot) \in H$, such that

$$\hat{x}|x \rangle = x|x \rangle, \quad (21)$$

$x \in \mathbb{R}$. Applying Proposition 1 we shall also prove the following statement.

Theorem 2. Suppose that $\rho \in \mathfrak{S}(H \otimes K)$ has the form

$$\rho = \int_{\mathbb{R}^2} \lambda(x, y) |x\rangle\langle y| \otimes |h_x\rangle\langle h_y| dx dy, \quad (22)$$

with unit vectors $h_x \in K$ and a positive definite function $\lambda(x, y)$. Then,

$$S((Id \otimes \Omega)(\rho)) \geq S(\rho) + \int_{\mathbb{R}} \pi(x) S(\Omega(|h_x\rangle\langle h_x|)) dx,$$

where

$$\pi(x) = \lambda(x, x),$$

and Ω is an arbitrary quantum channel.

Proof. Suppose that a state $\rho \in \mathfrak{S}(L^2(\mathbb{R}) \otimes K)$ has the form (22). Let us define an orthogonal projection P by the formula

$$P = \int_{\mathbb{R}} |x\rangle\langle x| \otimes |h_x\rangle\langle h_x| dx. \quad (23)$$

It is straightforward to check that

$$\rho P = P \rho = \rho.$$

Therefore,

$$\text{supp} \rho \subset \text{supp} P.$$

Applying Proposition 1 to the state ρ and the channel $\Phi = Id \otimes \Omega$ we obtain

$$S((Id \otimes \Omega)(\rho)) - S(\rho) \geq -\text{Tr}((Id \otimes \Omega)(\rho) \log(Id \otimes \Omega)(P)).$$

Notice that using (22) and (23) the r.h.s. above results in

$$\begin{aligned} & \text{Tr}((Id \otimes \Omega)(\rho) \log(Id \otimes \Omega)(P)) = \\ & \text{Tr} \left(\int_{\mathbb{R}^2} \lambda(x, y) |x\rangle\langle y| \otimes \Omega(|h_x\rangle\langle h_y|) dx dy \int_{\mathbb{R}} |z\rangle\langle z| \otimes \log(\Omega(|h_z\rangle\langle h_z|)) dz \right) = \\ & \text{Tr} \left(\int_{\mathbb{R}} \lambda(x, x) \Omega(|h_x\rangle\langle h_x|) \log(\Omega(|h_x\rangle\langle h_x|)) dx \right). \end{aligned}$$

Corollary 3. Suppose that $\rho \in \mathfrak{S}(H \otimes K)$ has the form

$$\rho = \int_{\mathbb{R}^2} \lambda(x, y) |x \rangle \langle y| \otimes |h_x \rangle \langle h_y| dx dy,$$

with unit vectors $h_x \in K$ and a positive definite function $\lambda(x, y)$. Then,

$$S((Id \otimes \Omega)(\rho)) - S(\rho) \geq S_\Omega(Tr_H(\rho)),$$

where Ω is an arbitrary quantum channel.

Proof. It immediately follows from the inequality

$$\int_{\mathbb{R}} \pi(x) S(\Omega(|h_x \rangle \langle h_x|)) dx \geq S_\Omega(\sigma),$$

where $\sigma = \int_{\mathbb{R}} \pi(x) |h_x \rangle \langle h_x| dx$.

Corollary 4. The relation (12) is satisfied for the generalised dephasing channel Φ acting as

$$\Phi(|x \rangle \langle y|) = \lambda(x, y) |x \rangle \langle y|,$$

and an arbitrary channel Ω .

Proof. Given a unit vector $e \in L^2(\mathbb{R}) \otimes K$ there exists a measurable function $x \rightarrow h_x$ acting from the real line \mathbb{R} to unit vectors $h_x \in K$ and the function $\nu(x)$, $\int_{\mathbb{R}} |\nu(x)|^2 dx = 1$ such that

$$|e \rangle \langle e| = \int_{\mathbb{R}^2} \nu(x) \bar{\nu}(y) |x \rangle \langle y| \otimes |h_x \rangle \langle h_y| dx dy. \quad (24)$$

Applying the generalised dephasing channel Φ to (24) we get the state

$$\rho = (\Phi \otimes Id)(|e \rangle \langle e|) = \int_{\mathbb{R}^2} \lambda(x, y) \nu(x) \bar{\nu}(y) |x \rangle \langle y| \otimes |h_x \rangle \langle h_y| dx dy. \quad (25)$$

It satisfies the condition of Theorem 2 if one replace $\lambda(x, y)$ by $\lambda(x, y) \nu(x) \bar{\nu}(y)$.

3 Quantum phase-damping channels

Recall that a quantum dephasing channel Φ , defined by the formula (7), is said to be a phase damping channel if the condition (13) is satisfied. Analogously, starting from a generalised dephasing channel Φ defined by (10), we can introduce a generalised phase damping channel if the following condition is satisfied

$$\lambda(x, y) = \lambda(x - y), \quad \lambda(-x) = \overline{\lambda}(x). \quad (26)$$

Since the kernels λ_{nm} in (7) and $\lambda(x, y)$ in (10) are positive definite, then so are the functions λ_n and $\lambda(x)$, that is

$$\sum_{n,m} \lambda_{n-m} c_n \bar{c}_m \geq 0$$

and

$$\sum_{n,m} \lambda(x_n - x_m) c_n \bar{c}_m \geq 0$$

for any choice of $c_n \in \mathbb{C}$ and $x_n \in \mathbb{R}$, $1 \leq n \leq N < +\infty$. Thus, to classify all phase-damping channels we can apply the following theorem.

Theorem 3 (Bochner's theorem [9]). *Suppose that f is a positive definite function on a locally compact Abelian group G normalised by the condition $f(e) = 1$. Then, there exists a unique probability measure μ on the dual group \hat{G} such that*

$$f(g) = \int_{\hat{G}} \langle \hat{h}, g \rangle d\mu(\hat{h}).$$

We shall consider three possible cases corresponding to $G = \mathbb{Z}_N$, \mathbb{Z} and \mathbb{R} .

3.1 Finite dimension

Suppose that the Hilbert space H has a finite dimension, $\dim H = N < +\infty$. According to (7) $\Phi(|e_n\rangle\langle e_m|) = \lambda_{m-n} |e_n\rangle\langle e_m|$, where $0 \leq n, m < N$. Then the following theorem holds true.

Theorem 4. *The complex numbers (λ_n) are the discrete Fourier transform of a probability distribution $(\pi_n)_{n=0}^{N-1}$ determined by the formula*

$$\lambda_n = \sum_{k=0}^{N-1} \exp\left(\frac{2\pi n k i}{N}\right) \pi_k.$$

Moreover, there exists a unitary operator $U : H \rightarrow H$ and an orthonormal basis (f_n) in H such that

$$Uf_n = f_{n+1 \bmod N},$$

$0 \leq n \leq N-1$, and

$$\Phi(\rho) = \sum_{n=0}^{N-1} \pi_n U^n \rho U^{*n},$$

$\rho \in \mathfrak{S}(H)$.

Proof. Here it is more convenient to give a direct proof without using Bochner's theorem. Let us consider the operator T acting in \mathbb{C}^N by the formula

$$(T\nu)_m = \sum_{n=0}^{N-1} \lambda_{m-n} \nu_n,$$

where $\nu = (\nu_0, \dots, \nu_{N-1}) \in \mathbb{C}^N$. Since λ_n is a positive definite function we get

$$(\nu, T\nu) \geq 0. \quad (27)$$

Taking into account the Parseval equality for the discrete Fourier transform we obtain that (27) results in

$$\sum_{n,m=0}^{N-1} \pi_n |\nu_n|^2 \geq 0,$$

for any choice of complex numbers (ν_n) . It implies that $\pi_n \geq 0$. On the other hand,

$$\sum_{n=0}^{N-1} \pi_n = \lambda_0 = 1.$$

Then consider the unitary operator U acting in H as follows

$$U|e_n\rangle = \exp\left(\frac{2\pi ni}{N}\right)|e_n\rangle, \quad (28)$$

$0 \leq n < N$. Let us define the orthonormal basis (f_n) in H by the formula

$$|f_n\rangle = \sum_{m=0}^{N-1} \exp\left(\frac{2\pi nmi}{N}\right)|e_m\rangle.$$

Applying the operator (28) to vectors (f_n) we get

$$U|f_n\rangle = \sum_{m=0}^{N-1} \exp\left(\frac{2\pi(n+1)mi}{N}\right)|e_m\rangle = |f_n\rangle.$$

3.2 Infinite dimension

Now let $\dim H = +\infty$. Fix the orthonormal basis (e_n) and consider a quantum phase damping channel Φ defined by (2).

Theorem 5. *The complex numbers (λ_n) are the Fourier transform of a probability measure μ on the unit circle \mathbb{T} such that*

$$\lambda_n = \int_{\mathbb{T}} \exp(2\pi n t i) d\mu(t).$$

Moreover, there exists a unitary representation $t \rightarrow U_t$ of the multiplicative group \mathbb{T} in H such that

$$\Phi(\rho) = \int_{\mathbb{T}} U_t \rho U_t^* d\mu(t),$$

$\rho \in \mathfrak{S}(H)$.

Proof. Due to Bochner's theorem there exists a probability measure μ such that

$$\lambda_n = \int_{\mathbb{T}} \exp(2\pi n t i) d\mu(t).$$

Let us define a unitary representation $\mathbb{T} \ni t \rightarrow U_t$ by the formula

$$U_t |e_n\rangle = e^{2\pi n t i} |e_n\rangle,$$

$n \in \mathbb{Z}$. Consider a quantum channel $\tilde{\Phi}$ of the following form

$$\tilde{\Phi}(\rho) = \int_{\mathbb{T}} U_t \rho U_t^* d\mu(t),$$

$\rho \in \mathfrak{S}(H)$. It follows that

$$\tilde{\Phi}(|e_n\rangle\langle e_m|) = \int_{\mathbb{T}} e^{2\pi(n-m)t i} d\mu(t) = \lambda_{n-m} |e_n\rangle\langle e_m|.$$

Hence $\tilde{\Phi} = \Phi$.

3.3 The generalised phase-damping channel

Let us consider here the case $H = L^2(\mathbb{R})$. There the generalised quantum phase damping channel is defined by the formula

$$\Phi(|x\rangle\langle y|) = \lambda(x - y)|x\rangle\langle y|. \quad (29)$$

Theorem 6. *The function $\lambda(x)$ in (29) is the Fourier transform of a probability measure μ on the line \mathbb{R} defined by the formula*

$$\lambda(x) = \int_{\mathbb{R}} \exp(ixy) d\mu(y).$$

Moreover, there exists a strong continuous one-parameter group of unitaries $t \rightarrow U_t$, $U_0 = I$, such that

$$\Phi(\rho) = \int_{\mathbb{R}} U_t \rho U_t^* d\mu(t),$$

$\rho \in \mathfrak{S}(H)$.

Proof. Due to Bochner's theorem there exists a probability measure μ on the line \mathbb{R} such that

$$\lambda(x) = \int_{\mathbb{R}} \exp(ixy) d\mu(y).$$

Let us define a strong continuous group of unitaries (U_t) by the formula

$$(U_t \psi)(x) = e^{itx} \psi(x), \quad \psi \in H.$$

Consider a quantum channel $\tilde{\Phi}$ of the following form

$$\tilde{\Phi}(\rho) = \int_{\mathbb{R}} U_t \rho U_t^* d\mu(t),$$

$\rho \in \mathfrak{S}(H)$. It follows that for $\psi, \phi, \xi \in H$ we get

$$(\tilde{\Phi}(|\psi\rangle\langle\phi|)\xi)(x) = \int_{\mathbb{R}^2} e^{it(x-y)} \psi(x) \bar{\phi}(y) \xi(y) dy d\mu(t) = \int_{\mathbb{R}} \lambda(x-y) \psi(x) \bar{\phi}(y) \xi(y) dy.$$

It implies that $\tilde{\Phi} = \Phi$.

4 Conclusion

We have derived a nontrivial lower bound on the entropy gain with respect to the action of an arbitrary quantum channel affecting only one part of the system, for a class of bipartite quantum states (Corollary 1 and 3). Based on this result we have estimated (from below) the output entropy for tensor product of the dephasing quantum channel and an arbitrary channel (Corollary 2 and 4). Finally, we have introduced a classification of quantum phase-damping channels resulting as special cases of the dephasing channels (Theorem 4, 5 and 6).

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